allows us to make the formal replacement of the sum over k by an integral

$$\sum_k \to \frac{V}{(2\pi c)^3} \int_0^\infty d\omega_k \, \omega_k^2 \int_0^\pi d\theta \, \sin\theta \int_0^{2\pi} d\phi. \tag{9.44}$$

All the integrations in the above equation can be performed analytically. Since

$$\int_0^{2\pi} d\phi \sin^2\phi = \int_0^{2\pi} d\phi \cos^2\phi = \pi, \tag{9.45}$$

and

$$\int_0^\pi d\theta \left(1 + \cos^2\theta\right) \sin\theta = \frac{8}{3}, \tag{9.46}$$

we obtain for the damping rate

$$\Gamma = \frac{1}{4\pi\varepsilon_0} \frac{4\mu^2\omega_0^3}{3\hbar c^3}, \tag{9.47}$$

which is the Einstein's A coefficient for spontaneous emission. The damping rate is given in terms of the atomic parameters, which comes from a fully quantum treatment of the atom–field interaction.

9.5 The Bloch–Siegert Shift: An Example of Non-RWA Effects

In Chapter 2, we showed that the exact interaction Hamiltonian between a two-level atom and an EM field contains the energy non-conserving terms called the counter-rotating terms. These terms are usually ignored as being rapidly oscillating over the time scale $t \sim 1/\omega_0$, and an obvious question arises whether there are situations where these terms could generate physical observable phenomena. In this section, we discuss the effect of the counter-rotating terms on spontaneous emission from a two-level atom coupled to a vacuum field. The counter-rotating terms are included into the interaction by not making the RWA on the interaction Hamiltonian between the atom and the vacuum field. As we shall see, the counter-rotating terms can produce a small shift of the atomic levels, known in the literature as the Bloch–Siegert shift.

The exact interaction Hamiltonian that includes the counter-rotating terms is of the form

$$\hat{H}_{\text{int}} = -\frac{1}{2}i\hbar\sum_{k} g_k\left[S^{+}\hat{a}_k(t) - S^{-}\hat{a}_k^{\dagger}(t) + S^{-}\hat{a}_k(t) - S^{+}\hat{a}_k^{\dagger}(t)\right]. \tag{9.48}$$

In addressing the question of the role of the counter-rotating terms, we derive, with the procedure outlined in Section 9.3, the master equation for the reduced density operator of the atom. The derivation, the details of which are left for the reader as a tutorial exercise, shows that the counter-rotating terms, appearing in the Hamiltonian (9.48), lead to additional terms in the master equation which takes the form

$$\begin{aligned}\frac{\partial\rho}{\partial t} = &-i\left(\omega_0+\Delta\right)\left[S^{+}S^{-},\rho\right] - i\Delta\left(S^{+}\rho S^{+} - S^{-}\rho S^{-}\right)\\ &-\frac{1}{2}\Gamma\left\{S^{+}S^{-}\rho + \rho S^{+}S^{-} - 2S^{-}\rho S^{+}\right.\\ &\left.-2S^{+}\rho S^{+} - 2S^{-}\rho S^{-}\right\}.\end{aligned} \tag{9.49}$$

There are no terms present like $S^{+}S^{+}\rho$ or $S^{-}S^{-}\rho$, since $S^{+}S^{+} = S^{-}S^{-} \equiv 0$.

An important modification of the master equation is the appearance of additional terms of the form $S^{+}\rho S^{+}$ and $S^{-}\rho S^{-}$, which indicates a two-photon nature of the counter-rotating terms. In the following, we will ignore the effect of the additional terms on the small Lamb shift, and we will check how the two extra terms in the dissipative part of the master equation modify the spontaneous emission.

To identify the role of the counter-rotating terms, we consider the evolution of the atomic dipole moment (coherence). Using the master equation (9.49), we obtain two coupled differential equations for the off-diagonal density matrix elements

$$\begin{aligned}\dot{\rho}_{12} &= i\omega_0\rho_{12} - \frac{1}{2}\Gamma\rho_{12} + \Gamma\rho_{21},\\ \dot{\rho}_{21} &= -i\omega_0\rho_{21} - \frac{1}{2}\Gamma\rho_{21} + \Gamma\rho_{12}.\end{aligned} \tag{9.50}$$

Thus, the additional terms brought by the counter-rotating terms couple the coherencies ρ_{12} to its conjugate ρ_{21}. This is the modification of the dynamics of the atoms due to the presence of the counter-rotating terms.[a]

We can solve the set of the coupled differential equations using, for example, the Laplace transform method, which allows us to transform the differential equations into a set of two coupled algebraic equations. We can write the set of the transformed equations in a matrix form as

$$\begin{pmatrix} \left(z + \frac{\Gamma}{2} - i\omega_0\right) & -\Gamma \\ -\Gamma & \left(z + \frac{\Gamma}{2} + i\omega_0\right) \end{pmatrix} \begin{pmatrix} \rho_{12}(z) \\ \rho_{21}(z) \end{pmatrix} = \begin{pmatrix} \rho_{12}(0) \\ \rho_{21}(0) \end{pmatrix}. \tag{9.51}$$

According to the Laplace transform method, the time evolution of the atomic coherence is determined by the roots of the determinant of the 2×2 matrix. The determinant is of the form

$$\begin{aligned} D(z) &= \left(z + \frac{\Gamma}{2}\right)^2 + \omega_0^2 - \Gamma^2 = \left(z + \frac{\Gamma}{2}\right)^2 + \omega_0^2\left(1 - \frac{\Gamma^2}{\omega_0^2}\right) \\ &= \left[z + \frac{\Gamma}{2} + i\left(\omega_0 - \frac{\Gamma^2}{2\omega_0}\right)\right]\left[z + \frac{\Gamma}{2} - i\left(\omega_0 - \frac{\Gamma^2}{2\omega_0}\right)\right]. \end{aligned} \tag{9.52}$$

The roots of the polynomial $D(z)$ determine the time evolution of the atomic coherence such that the real parts of the roots contribute damping rates, while the imaginary parts contribute frequencies of the oscillations. According to the expression (9.52), the counter-rotating terms contribute to the imaginary parts $\left(\omega_0 - \Gamma^2/2\omega_0\right)$, that is, they give rise to a shift of the atomic resonance by the amount of $\Gamma^2/2\omega_0$. We identify this shift with the spontaneous emission Bloch–Siegert shift. Since typically $\Gamma \ll \omega_0$, it is apparent that the shift is very small.

In summary of this section, we may state that the counter-rotating terms (the energy non-conserving terms) can have a physical effect on the atomic dynamics. For an atom interacting with a multi-mode reservoir, the terms cause a small shift of the atomic levels, known as the Bloch–Siegert shift.

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[a]The coupling of the atomic coherence to its conjugate is formally similar to that appearing in the equations of motion for a two-level atom interacting with a squeezed vacuum.